AN INTRODUCTION TO THE THEORY OF LINEAR ALGEBRAIC GROUPS.

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II. Linear Algebraic Groups.

1. Definitions

Here an algebraically closed field K is fixed. All algebraic varieties are defined over K.

Definition 1.1. *An algebraic group is an algebraic variety G which is also a group such that the map*

$$\mu : G \times G \to G$$

defined by the formula $\mu((x, y)) = xy$ and the map

$$i : G \to G$$

given by the formula $i(x) = x^{-1}$ are morphisms of algebraic varieties (here $G \times G$ is considered as a product of varieties). If the variety G is affine then G is called an affine algebraic group (or simple an affine group).

Remark 1.2. *An algebraic group is not a topological group with respect to Zariski topology. In the definition of topological group the topology on $G \times G$ should be a topology of products of topological space .*

Examples

1. Additive group $\mathbf{G}_a$

$\mathbf{G}_a = A^1_K$ is the one-dimensional affine space and the operation : $x + y$. Obviously, $\mu((x, y)) = x + y$ and $i(x) = -x$ are morphisms of affine varieties. Here $K[\mathbf{G}_a] = K[x]$.

2. General linear group $\mathbf{GL}_n$

Let $M = M_n(K)$ be a set of all $n \times n$ matrices over K. We consider it as the affine space $A^{n^2}_K$ with double numeration of the basis $\{x_{ij}\}$ of the the space of linear functions. We consider the subset of invertible matrices in M as an open set $U_M(\det)$ (here $\det \in K[M]$ is the determinant function). Note, that $U_M(\det)$ is an affine variety (it can be embedded $U_M(\det) \hookrightarrow A_K^{n^2+1}$ as a closed subset) and its affine algebra is equal to $K[x_{ij}, \det^{-1}]$. We denote this affine variety $\mathbf{GL}_n$. Define $\mu((g_1, g_2)) = g_1 g_2$ (products of matrices). Then μ, i are morphisms. Thus, $\mathbf{GL}_n$ is an algebraic group.

In the case $n = 1$ we have $M = M_1(K) = A_K^1 = K$ and $K[M] = K[x]$. We put $\det = x$. Thus, $K[\mathbf{CL}_1] = K[x, x^{-1}]$ and $\mathbf{GL}_1$ is an open set $A_K^1 \setminus \{0\} = K^*$. The group $\mathbf{GL}_1$ is called also the multiplicative group or one-dimensional torus and it is also denoted by the symbol $\mathbf{G}_m$.

3. Special linear group $\mathbf{SL}_n$

Let $M = M_n(K)$ be a set of all $n \times n$ matrices over K. Let $\mathbf{SL}_n = V_M((\det - 1))$ be a closed subset of M (i.e all matrices with determinant 1). Then $K[\mathbf{SL}_n] = K[\{x_{ij}\}/(\det - 1)$. If we put $\mu((g_1, g_2) = g_1 g_2$ (products of matrices) we again get an algebraic group.

Exercises.

Prove that the following subgroups of $GL_n(K)$ can be considered as algebraic groups:

1) finite subgroups;

2) the group $\mathbf{D}_n$ of diagonal non-singular matrices;

3) the group $\mathbf{T}_n$ of non-singular upper triangular matrices;

4) the group $\mathbf{U}_n$ of unipotent upper triangular matrices;

5) the group $\mathbf{O}_n$ of orthogonal matrices ($O_n(K) = \{X \in GL_n(K) \mid XX^T = 1\}$);

6) the special orthogonal group $\mathbf{SO}_n$ (the group of orthogonal matrices with determinant one)

7) the symplectic group $\mathbf{Sp}_{2n}$

$$Sp_{2n} = \{X \in GL_{2n}(K) \quad | \quad X^T \begin{pmatrix} \mathbf{0} & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{0} \end{pmatrix} X = \begin{pmatrix} \mathbf{0} & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{0} \end{pmatrix}\}$$

In the previous examples we have dealt with groups whose varieties are affine varieties. Now we give an example of algebraic group which is not an affine variety.

Elliptic curves and abelian variety

In this example char $K \neq 2, 3$. Let P_K^2 be the projective plane and let G be the curve

$$x_0 x_2^2 = x_1^3 + a x_1 x_0^2 + b x_0^3 \quad (1)$$

where (x_0, x_1, x_2) are coordinates in P_K^2 and $a, b \in K$ such that polynomial has no $T^3 + aT + b$ multiple roots.

Let $L : c_0 x_0 + c_1 x_1 + c_2 x_2 = 0$ is a line on P_K^2. Then

$$L \cap G = \{A, B, C\}$$

(possibly, there is a coincidence of some points here). Put

$$A + B + C = E = (0, 0, 1).$$

One can check that this construction defines a structure of algebraic group on G where the point E is a neutral element. But the direct verification is quite long and clumsy. There is a more general geometrical approach (see, [Har], II, §6, Example 6.10.2).

Here we presented an example of an algebraic group which is a projective curve. Every projective non-singular curve X has an numerical invariant $g(X) \in \mathbb{N} \cup \{0\}$ which is called the genus of X. For every complete curve X it is possible to construct an algebraic group $J(X)$ of the dimension $g(X)$ such that $J(X)$ is a projective variety. It is always an abelian group. The variety $J(X)$ is called *the Jacobian of the curve* X. If $g(X) = 1$ and X is non-singular then X can be embedded in P_K^2 and in appropriate coordinate it has the equation (1). Non-singular curves of genus one are called *elliptic curves*. Thus, the Jacobian for elliptic curves can be identified with the same curves and the operation is given by intersections of the curve and lines in P_K^2.

A variety X is called *a complete variety* if for every variety Y the projection $X \times Y \to X$ is a closed map,i.e. the image of every closed subset is a closed subset. All projective varieties are complete. However, there are non-singular complete varieties which are not projective ([Har], II, §4).

Every algebraic group which is a complete variety is always an abelian group. The complete algebraic groups are called *abelian varieties*. By A. Weyl Theorem all abelian varieties are projective ([Har], II, §4). According C.Chevalley Theorem, every algebraic group G contains an affine normal subgroup $H \lhd G$ such that the factor-group is an abelian variety.

The theory of abelian varieties is a part of algebraic geometry (see, [Ma]). In this course we suppose to study only affine algebraic groups.

Below by an algebraic group we always mean a linear (affine) algebraic group

Definition 1.3. *A homomorphism of algebraic groups $G \to H$ is a group homomorphism which is also a morphism of varieties.*

Example The determinant map

$$\det : \mathbf{GL}_n \to \mathbf{GL}_1 = \mathbf{G}_m$$

is a homomorphism of algebraic groups.

Remark 1.4. *A closed subgroup H of algebraic group G is also algebraic group and the imbedding $H \hookrightarrow G$ is a homomorphism of groups (it easily follows from the definition).*

Definition 1.5. *Let G be an algebraic group and let F be a subfield of K. If the variety G is F-defined we say that G is an F-group and $G(F)$ be a group of F-points.*

2. Some elementary properties

Theorem 2.1. *Let G be an algebraic group. Then the variety G is smooth.*

Proof. Let $g \in G$ and let

$$f_g : G \to G$$

be the map given by the formula

$$f_g(x) = gx.$$

Since f_g is a composition of morphisms $j_g : X \to X \times Y$ given by the formula $j_g(x) = (x, g)$ and the morphism μ the map f_g is also a morphism. Since $f_g \circ f_{g^{-1}} = f_{g^{-1}} \circ f_g = id_G$ the morphism f_g is an isomorphism of variety G on itself. Hence

$$O_{g,G} \approx O_{e,G}$$

for every $g \in G$ (I, Section 2.**The ring of regular functions $O_{x,X}$ at the point x of an affine variety X**.. Exercise 1.). Since for every variety we can find an open smooth subset (II, Remark 2.35) every point of G is smooth.

□

Theorem 2.2. *Let G be an algebraic group and $e \in G$ be the identity element. Then:*

i. there exists only one irreducible component G^0 of G that contains e and this component is a closed normal subgroup of finite index.

ii. G^0 is a unique connected component containing e;

iii. any closed subgroup of G of finite index contains G^0.

Proof. i. Let X, Y be irreducible components of G containing e Then $X \times Y$ is irreducible (I, Proposition 2.14) and the closure $\overline{\mu(XY)} = \overline{XY}$ is irreducible (I, Proposition 2.15). Since $X, Y \subset XY$ we have

$$\overline{XY} = X = Y = XY.$$

Thus X is closed under the multiplication. Since i is a homeomorphism $i(X)$ is also an irreducible component and since $e \in i(X)$ we have $i(X) = X$ (because, as we see above, components which contain e should coincide). Hence X is a closed subgroup of G.

Now let $g \in G$. Then the map

$$\phi_g : G \to G$$

$\phi_g(x) = gxg^{-1}$ is an isomorphism of G on itself as an algebraic group (note, that multiplication $x \to gx$, $x \to xg^{-1}$ are isomorphisms of variety G on itself). Hence gXg^{-1} is an irreducible component of G containing e and therefore $G^0 \stackrel{def}{=} X$ is normal subgroup.

Further,

$$G = \bigcup_{g \in G} gG^0$$

and for every $g \in G$ the set gG^0 is an irreducible component of G. Since the number of irreducible components is finite we have

$$[G : G^0] \leq \infty.$$

ii. Since all irreducible components have the form gG^0 they are disjoint and therefore the irreducible components are connected components.

iii. Let $H \leq G$ be a closed subgroup of finite index. Then H^0 is a closed subgroup of finite index in G^0. Then H^0 is open in G^0. Then $H^0 = G^0$.

□

Definition 2.3. *The irreducible component G^0 containing the identity element is called the identity component of the group G. The group G is called connected if $G = G^0$.*

Remark 2.4. *All irreducible components of G have the form gG^0 and therefore they are isomorphic to each other as algebraic varieties. In particular, they have the same dimension.*

Exercise

1. Prove: GL_n, SL_n, T_n, U_n are connected but O_n is not ($char K \neq 2$).
2. Prove: if G is connected and $H \lhd G, \mid H \mid < \infty$ then $H \leq Z(G)$ (here $Z(G)$ is the center of G).

Theorem 2.5. *Let G be an algebraic group and let H be a subgroup (as an abstract group). Then*
i) $\overline{H}$ is a closed subgroup of G;
ii) if H contains an open set of $\overline{H}$ then $H = \overline{H}$

Proof.
i.

Lemma 2.6. *Let Γ be an algebraic group and let $X, Y \subset G$. Then*

$$\overline{X}\ \overline{Y} \subset \overline{XY}.$$

Proof. Put

$$M = \mu^{-1}(\overline{XY}).$$

Then M is a closed subset $G \times G$. We have

$$X \times Y \subset M \Rightarrow \overline{X \times Y} \subset M.$$

Now it is enough to prove

$$Z' = \overline{X \times Y} = \overline{X} \times \overline{Y} = Z.$$

We have

$$\forall x \in X : (x, Y) \subset \overline{X \times Y} \Rightarrow (X \times \overline{Y}) = \bigcup_{x \in X} \overline{(x, Y)} = \bigcup_{x \in X} (x, \overline{Y}) \subset$$
$$\subset \overline{X \times Y} = Z'.$$

Further,

$$\forall y \in \overline{Y} : (X, y) \subset \overline{X \times Y} \Rightarrow (\overline{X} \times \overline{Y}) = \bigcup_{y \in \overline{Y}} \overline{(X, y)} = \bigcup_{y \in \overline{Y}} (\overline{X}, y) \subset$$

$$\subset \overline{X \times Y} = Z'. \tag{3}$$

□

Lemma 2.7. *Let Γ be an algebraic group and let U, V be two dense open subsets of Γ. Then $UV = G$.*

Proof. Let $V^{-1} = \{v^{-1} \mid v \in V\}$. Then V^{-1} is a dense open subset of G and therefore for every $g \in G$

$$U \cap gV^{-1} \neq \emptyset \Rightarrow u = gv^{-1} \text{ for some } u \in U, v \in V \Rightarrow g = uv.$$

□

i. From Lemma 2.6, $\overline{H}\ \overline{H} \subset \overline{H}$. Hence $\overline{H}$ is closed under the operation on G. Further,

$$\overline{H}^{\,-1} \supset H^{-1} \Rightarrow \overline{H}^{\,-1} \supset \overline{H^{-1}} = \overline{H}.$$

By the same arguments we have the inverse inclusion and therefore $\overline{H}^{\,-1} = \overline{H}$.

ii. Let $U \subset H$ be a dense open subset of $\overline{H}$. Then, by Lemma 2.7, $\overline{H} = UU \subset HH = H$. □

Theorem 2.8. *Let G be an algebraic group and let $\{H_\alpha\}_{\alpha \in \mathfrak{A}}$ be a set of constructible subsets such that $e \in H_\alpha$ and $\overline{H}_\alpha$ is an irreducible set for every α . Further, let H be a subgroup generated by all H_α. Then H is closed and connected. Moreover, there exists a finite number of sets $H_{\alpha_1}, \ldots, H_{\alpha_n}$ (possibly with $H_{\alpha_i} = H_{\alpha_j}$ for some $i \neq j$) such that*

$$H = H_{\alpha_1} H_{\alpha_2} \cdots H_{\alpha_n}.$$

Proof. Let $H_{\alpha_1}, \ldots, H_{\alpha_n}$ (possibly with $H_{\alpha_i} = H_{\alpha_j}$ for some $i \neq j$) . Consider the dominant morphism

$$\phi : \overline{H}_{\alpha_1} \times \cdots \times \overline{H}_{\alpha_n} \to \overline{H_{\alpha_1} \cdots H_{\alpha_n}}.$$

Since $\overline{H}_{\alpha_1}, \ldots, \overline{H}_{\alpha_n}$ are irreducible then $\overline{H_{\alpha_1} \cdots H_{\alpha_n}}$ is irreducible (I, Proposition 2.15). Put $M = H_{\alpha_1} \cdots H_{\alpha_n}$. If $M = H$ then the statement holds.

Suppose $\overline{M} = \overline{H}$. From the condition of the Theorem $H_\alpha = L_{\alpha_1} \cup L_{\alpha_2} \cup \cdots \cup L_{\alpha_d}$ where $L_{\alpha_i} = C_{\alpha_i} \cap U_{\alpha_i}$ for some closed subset C_{α_i} of G and some open subset U_{α_i} of G. Since

$$\overline{H_\alpha} = C_{\alpha_1} \cup C_{\alpha_2} \cdots \cup C_{\alpha_d}$$

is an irreducible set then $\overline{H}_\alpha = C_{\alpha_{i_0}}$ for some i_0. Hence every H_α contains a dense open subset V_α of $\overline{H}_\alpha$. Further, the subset

$$V = V_{\alpha_1} \times V_{\alpha_2} \times \cdots V_{\alpha_n} \subset H_{\alpha_1} \times \cdots \times H_{\alpha_n} \subset \overline{H}_{\alpha_1} \times \cdots \times \overline{H}_{\alpha_n}.$$

is open. There exists an open subset $U \subset \overline{H_{\alpha_1} \cdots H_{\alpha_n}}$ such that $\phi^{-1}(U) \subset \overline{H}_{\alpha_1} \times \cdots \times \overline{H}_{\alpha_n}$ (I, Proposition 2.20). Since ϕ is a continuous map there exists an open subset $U' \subset U$ of $\overline{M} = \overline{H_{\alpha_1} \cdots H_{\alpha_n}}$ such that $\phi^{-1}(U') \subset \phi^{-1}(U) \cap V$. Hence $U' \subset M$ is a dense open subset of $\overline{M}$. Since M contains a dense open subset of $\overline{M} = \overline{H}$ we have $M^2 = H = \overline{H}$ by Lemma 2.7.

Suppose $\overline{M} \neq \overline{H}$. Consider $N = H_\alpha M$ for some α. We have $\overline{M} \subset \overline{N}$. Suppose

$$\overline{N} = \overline{M}.$$

We have

$$\overline{M} \subset H_\alpha \overline{M} \subset \overline{N} = \overline{M} \Rightarrow H_\alpha \overline{M} = \overline{M} \tag{1}$$

If (1) holds for every α it holds for H (recall, that H is generated by elements of groups H_α). Then

$$H\overline{M} = \overline{M} \Rightarrow \overline{H}\ \overline{M} = \overline{M} \overset{M \subseteq H}{\Rightarrow} \overline{H} = \overline{M}$$

which is a contradiction with our assumption. Thus, we may assume $\overline{M} \subsetneq \overline{N}$ and therefore $\dim \overline{N} > \dim \overline{M}$ (I, Section 1. Exercise 2). Since the dimension of $\overline{H}$ is finite we can get M with $\overline{M} = \overline{H}$.

□

Corollary 2.9. *Let H_1, H_2 be two closed subgroup of an algebraic group. If H_1 is connected then the commutator subgroup $[H_1, H_2]$ is also connected.*

Proof. The group $[H_1, H_2]$ is generated by elements of irreducible constructible sets $\{h_1 h_2 h_1^{-1} h_2^{-1} \mid h_1 \in H_1\}_{h_2 \in H_2}$. □

Theorem 2.10. *Let $\alpha : G \to H$ be a morphism of algebraic groups. Then*

i. $\operatorname{Im}\alpha$ *is a closed subgroup of H;*
ii. $(\operatorname{Im}\alpha)^0 = \alpha(G^0)$*;*
iii. $\operatorname{Ker}\alpha$ *is a closed subgroup of G;*
iiii. $\dim \operatorname{Ker}\alpha + \dim \operatorname{Im}\alpha = \dim G$.

Proof.

i. It follows from Theorem 2.5.

ii. The subgroup $\alpha(G^0)$ of $\operatorname{Im}\alpha$ is connected and it has a finite index. Hence our statment follows from Theorem 2.2 (iii.)

iii. $\operatorname{Ker}\alpha = \alpha^{-1}(e)$;

iiii. There exists an open set $V \subset \operatorname{Im}\alpha$ such that for every $h \in V$ the dimension of the fiber $\alpha^{-1}(h)$ is equal to $\dim G - \dim \operatorname{Im}\alpha$ (I, Proposition 2.24.b). If $g \in \alpha^{-1}(h)$ then $\alpha^{-1}(h) = g \operatorname{Ker}\alpha$. Thus all fibers are isomorphic and therefore we get our assertion. □

3. Affine groups and affine algebras

Let G be an affine (linear) group and let $A = K[G]$ be the corresponding affine algebra. Then the affine algebra which corresponds to $G \times G$ is $A \otimes_K A$ (I,Proposition 2.14). The morphism of varieties

$$\mu : G \times G \to G$$

implies the homomorphism of algebras

$$\mu^* : A \to A \otimes_K A$$

which is called the *co-multiplication.* If $f \in A$ and $\mu^*(f) = \sum \phi_i \otimes \psi_i$ then

$$\mu^*(f)((x,y)) = f(\mu(x,y)) = f(xy) = \sum \phi_i(x)\psi_i(y).$$

The the isomorphism of varieties

$$i : G \to G$$

implies the isomorphism of algebras

$$i^* : A \to A.$$

which is called the *antipode.* Here

$$i^*(f)(x) = f(x^{-1}).$$

We introduce the homomorphism

$$e^* : A \to K$$

which correspond to the embedding $e \hookrightarrow G$. Here

$$e^*(f) = f(e).$$

Also, by

$$\rho : G \to G$$

denote the constant morphism $\rho(g) = e$.

Consider the axioms of groups. The associative law can be expressed by the following commutative diagram:

$$\begin{array}{ccc} G\times G\times G & \stackrel{\mu\times 1}{\to} & G\times G \\ 1\times\mu\downarrow & & \downarrow\mu \\ G\times G & \stackrel{\mu}{\to} & G. \end{array} \qquad (I)$$

Then we have a dual diagrams for algebras

$$\begin{array}{ccc} A\otimes A\otimes A & \stackrel{\mu^*\otimes 1}{\leftarrow} & A\times A \\ 1\otimes\mu^*\uparrow & & \uparrow\mu^* \\ A\otimes A & \stackrel{\mu^*}{\leftarrow} & A. \end{array} \qquad (I')$$

The axiom of the neutral element:

$$\begin{array}{ccc} G & \stackrel{\rho\times 1}{\to} & G\times G \\ 1\times\rho\downarrow & & \downarrow\mu \\ G\times G & \stackrel{\mu}{\to} & G \end{array} \qquad (II)$$

(here also $\stackrel{1}{\to}$ from left upper corner to lower right corner). The dual diagram

$$\begin{array}{ccc} A & \stackrel{\rho^*\otimes 1}{\leftarrow} & A\otimes A \\ 1\otimes\rho^*\uparrow & & \uparrow\mu^* \\ A\otimes A & \stackrel{\mu^*}{\leftarrow} & A \end{array} \qquad (II')$$

(here also $\stackrel{1}{\leftarrow}$ from lower right corner to left upper corner). Note,

$$\rho^* : A \stackrel{e^*}{\to} K \hookrightarrow A.$$

The existence of the inverse element:

$$\begin{array}{ccc} G & \stackrel{i\times 1}{\to} & G\times G \\ 1\times i\downarrow & & \downarrow\mu \\ G\times G & \stackrel{\mu}{\to} & G \end{array} \qquad (III)$$

(here also $\stackrel{\rho}{\to}$ from left upper corner to lower right corner)

The dual diagram:

$$
\begin{array}{ccc}
A & \overset{i^*\otimes 1}{\longleftarrow} & A\otimes A \\
1\otimes i^* \uparrow & & \uparrow \mu^* \\
A\otimes A & \overset{\mu^*}{\longleftarrow} & A
\end{array} \qquad (III')
$$

(here also $\overset{\rho^*}{\leftarrow}$ from lower right corner to left upper corner).

Examples-exercises.

1) $G = G_a$. Here $A = K[G_a] = K[x], A\otimes A = K[x,y], \mu^*(x) = x+y, i^*(x) = -x, e^*(x) = 0$;

2) $G = G_m$. Here $A = K[G_m] = K[x, x^{-1}], A\otimes A = K[x, y, x^{-1}, y{-}1], \mu^*(x) = xy, i^*(x) = x^{-1}, e^*(x) = 1$.

3) $G = GL_n$. Here $A = K[GL_n] = K[\{x_{ij}\}, \det(x_{ij})^{-1}], A\otimes A = K[\{x_{ij}\}, \{y_{ij}\}, \det(x_{ij})^{-1}, \det(y_{ij})^{-1}]$ and

$$\mu^*(x_{ij}) = \sum_k x_{ik}y_{kj},$$

$$i^*(x_{ij}) = \frac{A_{ji}}{\det(x_{ij})}$$

where A_{ji} is a compliment to x_{ji} and $e^*(x_{ij}) = \delta_{ij}$ (Kroneker symbol).

If we have an affine K-algebra A satisfying $I' - III'$ we get an affine (linear) algebraic group if we define dual morphisms on $\mathrm{Spec}_{\mathfrak{m}} A$.

If we consider any commutative algebra A over a ring K with homomorphisms μ^*, e^*, i^* satisfying $I' - III'$ we say that we have an affine group scheme.

Any associative algebra over a commutative ring K which has maps μ^*, e^* satisfying axioms I', II' is called *associative Hopf algebra with identity.* A Hopf algebra with anti-isomorphism i satisfying $I' - III'$ and such that the opposite algebra A^{op} (where the multiplication is defiend : $x * y = yx$) also satisfies $I' - III'$ for μ^*, e^*, i^{-1*} is called a *quantum group.*

4. G-spaces

Definition 4.1. *Let G be an algebraic group and X be a variety. The variety X is called G-space if the action of G on X is given and this action $G\times X \to X$ is a morphism of varieties. For a point $x\in X$ the subgroup $\{g\in G \mid g(x) = x\}$ is called the isotropy group of x or the stabilizer of x and denote G_x. The set $Gx = \{g(x) \mid g\in G\}$ is called the orbit of the point x and denote also by $O(x)$.*

Examples. 1) multiplication and conjugation, linear transformations (representations)

Proposition 4.2. *Let G be an algebraic group and let X be a G-space. Then*

i. for every $x \in X$ the group G_x is closed;
ii. for every $x \in X$ the orbit $O(x)$ is open in its closure $\overline{O(x)}$;
iii. there exists a closed orbit.

Proof. Let $\phi : G \to O(x)$ be a morphism: $\phi(g) = g(x)$ then $\phi^{-1}(x) = G_x$ (whence i.). There is an open set $U \in \overline{O(x)}$ such that $U \subset O(x)$. Hence $O(x) = \cup_{g \in G} gU$ (ii.). If $O(x) \subsetneqq \overline{O(x)}$ then $\overline{O(x)} \setminus O(x)$ is a G-space of the dimension $<$ than the dimension X (iii.) □

Examples.

1) Let $G = GL_n(K)$ act on itself by conjugation and let X be a set of unipotent matrices. Then each orbit $O(u),\ u \in X$ is uniquely defined by sizes $\lambda(u) = (n_1, n_2, \dots, n_k),\ \ n_1 \geq n_2 \geq \cdots \geq n_k,\ \ n_1 + n + 2 + \cdots + n_k = n$ of corresponding Jordan blocks. It is known:

$$O(v) \subset \overline{O(u)} \Leftrightarrow \lambda(v) \leq \lambda(u).$$

(Recall,

$$\lambda(v) = (n'_1, n'_2, \dots, n'_e) \leq \lambda(u)(n_1, n_2, \dots, n_k) \Leftrightarrow$$
$$\Leftrightarrow \sum_{i=1}^{s} n'_i \leq \sum_{i=1}^{s} n_i \ \text{ for every } \ s.)$$

(see, for instance, R. W. Carter, Finite groups of Lie type. Conjugacy Classes and Complex Characters.1985. John Wiley and Sons, Chichester, 1993. §13).

2) Let $B = T_n, G = B \times B, X = GL_n$ and let

$$(b_1 \times b_2)x = b_1 x b_2^{-1}.$$

Further, let $N \leq X$ be the group of monomial matrices and let $\phi : N \to S_n$ be the homomorphism corresponds to a monomial matrix its permutation. By W denote S_n and for $w \in W$ denote by $\dot{w}$ any fixed representative of $\phi^{-1}(w)$. Then every G-orbit has the form $G\dot{w}$, i.e.

$$X = GL_n(K) = \cup_{w \in W} B\dot{w}B = BWB$$

(Bruhat decomposition). Moreover,

$$B\dot{w}'B \subset \overline{B\dot{w}B} \Leftrightarrow w' \preccurlyeq w$$

(here $\preccurlyeq$ is the Bruhat order)

5. Rational representation. Affine groups = Linear groups

Definition 5.1. *A morphism $G \to GL_n$ of affine algebraic groups is called a rational representation of G.*

Theorem 5.2. *There exists an isomorphism of an affine algebraic group into a closed subgroup of GL_n for some n*

Proof. Let G be an affine algebraic group and let X be an affine G-space. The action $\alpha : G \times X \to X$ implies the homomorphism $\alpha^* : K[X] \to K[G] \otimes K[X]$.

Further, we have a natural linear representation

$$\rho : G \to GL(K[X])$$

$((gf)(x) = f(g^{-1}x))$.

Lemma 5.3. *Let $V \leq K[X]$ be a finitely dimensional linear subspace. Then*

i. there is a G-stable finitely dimensional subset $W \leq K[X]$ which contains V;

ii.the subspace V is G-stable $\Leftrightarrow \alpha^*(V) \subset K[G] \otimes V$;

iii. if V is G-stable then $\alpha_{|G\times V} : G \times V \to V$ is a rational representation.

Proof.

i. We may assume $\dim V = 1$. Then $V = Kf$. Let

$$\alpha^*(f) = \sum_i \phi_i \otimes \psi_i, \quad \phi_i \in K[G], \psi_i \in K[X].$$

Then

$$g(f)(x) = f(g^{-1}x) = \sum_i \phi_i(g^{-1})\psi_i(x) \Rightarrow Gf \subset \langle\{\psi_i\}\rangle \Rightarrow$$
$$\Rightarrow \dim(W = \langle Gf\rangle) \leq \infty.$$

ii.

$$\alpha^*(V) \subset K[G] \otimes V \Rightarrow gf \in V \text{ for every } f \in V, g \in G.$$

Suppose $V = \langle\{f_i\}\rangle$ is G-stable and let $\{f_i\} \cup \{\phi_\alpha\}$ be a basis of $K[X]$. Then for every $f \in V$

$$\alpha^*(f) = \sum_i \zeta_i \otimes f_i + \sum_\alpha \theta_\alpha \otimes \phi_\alpha \Rightarrow gf = \sum_i \zeta_i(g^{-1})f_i + \sum_\alpha \theta_\alpha(g^{-1})\phi_\alpha$$

$$\overset{GV=V}{\Rightarrow} \sum_\alpha \theta_\alpha(g^{-1})\phi_\alpha = 0 \text{ for every } g \overset{\{\phi_\alpha\} \text{ linearly independent}}{\Rightarrow} \theta_\alpha = 0$$

$$\text{for every } \alpha \Rightarrow \alpha^*(f) = \sum_i \zeta_i \otimes f_i \text{ for every } f \in V.$$

iii. Let

$$\alpha^*(f_i) = \sum_j \zeta_{ij} \otimes f_j.$$

Then for every $f = \sum_k t_i f_i,\ \ t_i \in K$ we have

$$gf = \sum_i t_i g f_i = \sum_i t_i (\sum_j \zeta_{ij}(g^{-1}) f_j) = \sum_{ij} t_i \zeta_{ij}(g^{-1}) f_j.$$

□

Now let $X = G$ and let G acts on itself by the right multiplication. Now $g(x) = xg^{-1}$ hence $gf(x) = f(g^{-1}(x)) = f(xg)$. Further, let $K[G] = K[f_1, \dots, f_n]$. By Lemma 5.2, we may assume $V = \langle f_1, \dots, f_n \rangle$ is G-stable and the rational representation

$$\rho : G \to GL(V)$$

is defined. Also, there exists a set of functions $\{\phi_{ij}\}, \phi_{ij} \in K[G]$ such that

$$\rho(g) f_i = \sum_j \phi_{ji}(g) f_j. \tag{1}$$

Thus we have a rational representation

$$\rho : g \to \begin{pmatrix} \phi_{11}(g) & \phi_{12}(g) & \cdots & \phi_{1n}(g) \\ \phi_{21}(g) & \phi_{22}(g) & \cdots & \phi_{2n}(g) \\ \cdots & \cdots & \cdots & \cdots \\ \phi_{n1}(g) & \phi_{2n}(g) & \cdots & \phi_{nn}(g) \end{pmatrix}$$

and the dual homomorphism

$$\rho^* : K[GL_n] = K[\{x_{ij}\}, \det(x_{ij})^{-1}] \to K[G]$$

which is defined by formulas

$$\rho^*(x_{ij})(g) = x_{ij}(\rho(g)) = \mu_{ij}(g),$$

$$\rho^*(\det(x_{ij})^{-1})(g) = \det(x_{ij})^{-1}\rho(g) = \det(\mu_{ij}(g)).$$

Hence

$$\rho^*(x_{ij})(g) = \mu_{ij},\ \ \rho^*(\det(x_{ij})^{-1}) = \det(\mu_{ij})^{-1}$$

and therefore,

$$\operatorname{Im} \rho^* = K[\{\mu_{ij}\}, \det(\mu_{ij})^{-1}]. \tag{2}$$

From (1) we have

$$\rho(g)f_i(e) = f_i(eg) = f_i(g) = \sum_j \mu_{ji}(g)f_j(e) \text{ for every } g \in G.$$

Whence

$$K[G] = K[f_1, \dots, f_n] \subset K[\{\mu_{ij}\}, \det(\mu_{ij})^{-1}]. \quad (3)$$

Now (2), (3) imply that ρ^* is a surjection. Further,

$$\operatorname{Ker}\rho^* = \{f \in K[GL_n] \mid f(\rho(g)) = 0 \text{ for every } g \in G\} = \mathfrak{I}_{GL_n}(\operatorname{Im}\rho) \Rightarrow$$

$$\Rightarrow \tilde{\rho}* : K[\operatorname{Im}\rho] = K[GL_n]/\mathfrak{I}_{GL_n}(\operatorname{Im}\rho) \to K[G] \text{ in an isomorphism of rings} \Rightarrow$$

$$\Rightarrow \operatorname{Im}\rho \approx G.$$

□

Remark 5.4. *Let G, H be two F- defined algebraic groups where $F \subset K$. Then we have two fixed F structure $K[G] = A \otimes K, K[H] = B \otimes K$. We say that a morphism $\phi : G \to H$ is F defined if $\phi^*(B) \subset A$. In this case, $\phi(G(F)) \subset H(F)$.If $\phi : G \to H$ is F defined morphism such that there exists an F-defined morphism $\phi^{-1} : H \to G$ then we say that G, H are F-isomorphic groups.*

If the affine algebraic group is F-defined it is F-isomorphic to a F-defined closed subgroup of GL_n ([Sp]*, Theorem 2.3.7).*

Theorem of Chevalley.

Theorem 5.5. *Let G be an algebraic group and let $H \leq G$ be a closed subgroup. There exists a rational representation*

$$r : G \to GL(V)$$

and a line $l \in V$ such that

$$H = \{g \in G \mid r(g)(l) = l\}.$$

Proof.

Lemma 5.6. *Let $\rho : G \to GL(K[G])$ be a linear representation induced by right translations of G on itself.*

$$\{g \in G \mid \rho(g)\mathfrak{I}_G(H) = \mathfrak{I}_G(H)\} = H.$$

Proof.

$$g, h \in H, f \in \mathfrak{I}_G(H) \Rightarrow \rho(g)f(h) = f(hg) = 0 \Rightarrow \rho(g)f \in \mathfrak{I}_G(H).$$

$$\forall \ f \in \mathfrak{I}_G(H), \rho(g)f \in \mathfrak{I}_G(H) \Rightarrow f(eg) = f(g) = 0 \Rightarrow g \in H.$$

□

Lemma 5.7. *Let $\rho : G \to GL(K[G])$ be a linear representation induced by right translations of G on itself. Then there exists a G-stable linear subset $V \leq K[G]$ and linear subset W such that*

$$H = \{g \in G \quad | \quad \rho(g)W = W\}.$$

Proof. Let $\mathfrak{I}_G(H) = (f_1, \dots, f_n)$. Let V be a G-stable subspace of $K[G]$ such that $f_1, \dots, f_n \in V$ (Lemma 5.3) and let $I = \mathfrak{I}_G(H)$. Put

$$W = V \cap I.$$

Now we get our statement from Lemma 5.6.

□

Lemma 5.8. *Let $W \leq V$ be subsets of $K[G]$ from the previous Lemma. Let $d = \dim W$ and let*

$$r : G \xrightarrow{\rho} GL(V) \xrightarrow{j} GL(\wedge^d V)$$

where j is the natural homomorphism. The r is a rational representation and

$$H = \{g \in G \quad | \quad r(g) \wedge^d W = \wedge^d W\}$$

Proof. The map r is a composition of the morphisms ρ and j. From Lemma 5.7.

$$H \subset \{g \in G \quad | \quad r(g) \wedge^d W = \wedge^d W\}.$$

Let

$$W = \langle v_1, \dots v_d \rangle, V = \langle v_1, \dots v_d, v_{d+1}, \dots v_m \rangle.$$

Let $g \in G, g \notin H$. The , by Lemma 5.7, $\rho(g)W \neq W$. We may assume

$$\rho(g)W = \langle v_{1+s}, \dots v_{d+s} \rangle$$

for some $s > 0$. Put

$$l = v_1 \wedge v_2 \wedge \dots \wedge v_d, \quad f = v_{1+s} \wedge v_{2+s} \wedge \dots \wedge v_{d+s}.$$

Then

$$r(g)l = \alpha f$$

for some $\alpha \in K^*$. Suppose $r(g)l = \beta l$ for some $\beta \in K^*$. Then $l = \gamma f$ for some $\gamma \in K^*$ but l, f are linearly independent in $\wedge^d V$.

□

□

Corollary 5.9. *Let G be an algebraic group an let H is a closed subgroup of G. Then there is a quasi-projective G-space X which consists of a single orbit and a point $x_0 \in X$ such that*

i.$G_{x_0} = H$;

ii.the fibers of the morphism $\psi : G \to X$ $(\psi(g) = g(x_0))$ are cosets gH.

Proof. X is the orbit of point l in $P(V)$. □

6. Jordan decomposition

Let V be a finitely dimension linear space over K and let $g \in GL(V)$. Then

$$g = g_s g_u = g_u g_s$$

where g_s is uniquely defined *semiseimple* (diagonalizible) linear operatotr and g_u is uniquely defined *unipotent* (a triagonalizible linear operator with all eigenvalues 1) linear operator is called the **Jordan decomposition** of g.

Definition 6.1. *Let V be an infinitely dimensional vector space. We say that $g \in \mathrm{End}(V)$ is locally finite if*

$$V = \bigcup_{\alpha} V_\alpha$$

where V_α is a finitely dimensional g-stable vector space for each α. A locally finite operator $g \in \mathrm{End}(V)$ is called semisimple (resp. unipotent) is the restriction of g on each V_α is semisimple (unipotent).

Suppose $g \in \mathrm{End}(V)$ is a locally finite operator. Then we have

$$g_\alpha \overset{def}{=} g_{|V_\alpha} = g_{\alpha s} g_{\alpha u}.$$

Since semisimple and unipotent components are uniquely defined by an operator we have

$$(g_{\alpha s})_{|V_\alpha \cap V_\beta} = (g_{\beta s})_{|V_\alpha \cap V_\beta}, \quad (g_{\alpha u})_{|V_\alpha \cap V_\beta} = (g_{\beta u})_{|V_\alpha \cap V_\beta}$$

for every α, β. Thus, the system of linear operators $\{g_{\alpha s}\}$ (respectively, $\{g_{\alpha u}\}$) defines a semisimple operator g_s (respectively, a unipotent operator g_u) such that $g = g_s g_u$ and such operators g_s, g_u are uniquely defined by g.

Now let G be an algebraic group and let $A = K[G]$. Then we have a linear representation

$$\rho : G \to GL(A)$$

$(\rho(g)(f)(x) = f(xg)$.

By Lemma 5.3, every element $\rho(g)$ is locally finite. Hnce we can define the Jordan decomposition

$$\rho(g) = \rho(g)_s \rho(g)_u.$$

Proposition 6.2. *There exist unique elements $g_s, g_u \in G$ such that $g = g_s g_u = g_u g_s$ and $\rho(g)_s = \rho(g_s), \rho(g)_u = \rho(g_u)$.*

Proof. We omit the proof of the following lemma from Linear Algebra.

Lemma 6.3. *i. Let $g \in \mathrm{End}(V), h \in \mathrm{End}(U)$ be two locally finite operators. Then $g \otimes h \in \mathrm{End}(V \otimes U)$ is locally finite operator and*

$$(g \otimes h) = (g_s \otimes h_s)(g_u \otimes h_u)$$

be its Jordan decomposition.

ii. Let $\phi : V \to U$ be a linear map. If $\phi \circ g = h \circ \phi$ then

$$\phi \circ g_s = h_s \circ \phi, \quad \phi \circ g_u = h_u \circ \phi.$$

iii. Let $W \leq V$ be a linear subspace. Then

$$g(W) = W \Rightarrow g_s(W) = W, g_u(W) = W.$$

Let

$$m : A \otimes A \to A$$

is a homomorphism defined by the formula

$$m(\sum_i \phi_i \otimes \psi_i) \overset{def}{=} \sum_i \phi_i \psi_i. \tag{1}$$

We have from (1)

$$m \circ (\rho(g) \otimes \rho(g)) = \rho(g) \circ m. \tag{2}$$

From (2) and Lemma 6.3

$$m \circ (\rho(g)_s \otimes \rho(g)_s) = \rho(g)_s \circ m. \tag{3}$$

Now (3) implies that $\rho(g)_s$ is an automorphism of A. Consider the map

$$\theta_{g_s} : A \to K$$

defined by the formula $\theta_{g_s}(f) = \rho(g)_s(f)(e)$. Since $\rho(g)_s$ is a homomorphism of A the map θ_{g_s} is a homomorphism of rings. Hence we have a morphism of varieties

$$\theta_{g_s*} : \{a\} = \operatorname{Spec}_{\mathfrak{m}} K \to \operatorname{Spec}_{\mathfrak{m}} A = G.$$

Put

$$g_s \stackrel{def}{=} \theta_{g_s*}(a).$$

Let us check

$$\rho(g_s) = \rho(g)_s. \tag{4}$$

We have

$$f(g_s) = \theta^*_{g_s*}(f) \stackrel{I,Sect.1,Ex.5}{=} \theta_{g_s}(f) = \rho(g)_s(f)(e). \tag{5}$$

If $\lambda : G \to GL(V)$ is a linear representation given by the formula $\lambda(g)(f)(x) = f(g^{-1}x)$ then $\lambda(g_1)\rho(g_2) = \rho(g_2)\lambda(g_1)$ for every $g_1, g_2 \in G$. Thus,

$$(\rho(g)_s(f))(x) = \lambda(x^{-1})(\rho(g)_s(f))(e) \stackrel{Lemma\ 6.3,ii.}{=}$$

$$= \rho(g)_s(\lambda(x^{-1})(f))(e) \stackrel{(5)}{=} \lambda(x^{-1})(f)(g_s) = f(xg_s) = \rho(g_s)f(x).$$

Whence (4).

Using the same arguments we can get $g_u \in G$ such that $\rho(g_u) = \rho(g)_u$.

Since ρ is faithful representation we have $g = g_s g_u = g_u g_s$. □

Theorem 6.4. *Let $\phi : G_1 \to G_2$ be a morphism of algebraic groups. Then for every $g \in G_1$ we have*

$$\phi(g)_s = \phi(g_s), \ \ \phi(g_u) = \phi(g)_u.$$

Proof. Consider the case $G_1 \leq G_2$ and $\phi : G_1 \hookrightarrow G_2$ is the natural embedding. Let $g \in G_1 \subset G_2$. By the previous theorem there exist $g_{s1}, g_{u1} \in G_1, g_{s2}, g_{u2} \in G_2$ such that

$$g = g_{s1}g_{u1} = g_{u1}g_{s1} = g_{s2}g_{u2} = g_{u2}g_{s2}$$

and

$$\rho_1(g)_s = g_{s1}, \rho_1(g)_u = g_{u1}, \quad \rho_2(g)_s = g_{s2}, \rho_2(g)_u = g_{u2}$$

where

$$\rho_1 : G_1 \to GL(K[G_1]), \quad \rho_2 : G_2 \to GL(K[G_2])$$

are the linear representations induced by right translations. By Lemma 5.6

$$G_1 = \{g \in G_2 \ \ | \ \ \rho(g)\mathfrak{I}_{G_2}(G_2) = \mathfrak{I}_{G_2}(G_2)\}.$$

By Lemma 6.3, iii.,

$$g_{s2}, g_{u2} \in G_1. \tag{6}$$

Further, the embedding $\phi : G_1 \hookrightarrow G_2$ implies the epimorphism

$$\phi^* : K[G_2] \to K[G_1]$$

and the corresponding homomorphism

$$r : \rho_2(G_1) \to GL(K[G_1]).$$

Obviously, $\rho_1 = r \circ \rho_2$ and the image with respect to r of any semisimple (unipotent) operator remains semisimple (unipotent). Hence, by (6) and the uniqueness of Jordan decomposition we get $g_{s1} = g_{s2}, g_{u1} = g_{u2}$.

Now let $\phi : G_1 \to G_2$ is a surjection. It is easy to see, that $K[G_2]$ is a g-stable subspace of $K[G_1]$ and $g_{|K[G_2]} = \phi(g)$. Let $f \in \phi^*(K[G_2])$. Then $f = \phi^*(h)$ for some $h \in K[G_2]$ Then for every $g \in G_1$ we have

$$\rho(g)f(x) = \rho(g)\phi^*(h)(x) = \phi^*(h)(xg) = h(\phi(x)\phi(g))$$

Put

$$h_1(y) \stackrel{def}{=} h(y\phi(g)) \quad \text{for every} \quad y \in G_2.$$

Then

$$\phi^*(h_1)(x) = h_1(\phi(x)) = h_1(\phi(x)\phi(g).$$

Hence $\phi^*(K[G_2])$ is $\rho(G_1)$-stable subset of $K[G_1]$. Since $\rho(g)\phi^*(h)(x) = h(\phi(x)\phi(g))$ we may identify $K[G_2]$ as a G-stable subset of $K[G_1]$ such that the restriction of $\rho(g)$ on $K[G_2]$ acts by right multiplication of arguments on $\phi(g)$. We also may identify the action of $\rho(g)$ with the action of g because ρ is an injective map.

According to Lemma 6. 3, iii., the subspace $K[G_2]$ is g_s and g_u-stable. But the restriction of a semisimple (unipotent) operator on stable subspace is again semisimple (unipotent). Hence $(g_s)_{|K[G_2]}, (g_s)_{|K[G_2]}$ are semisimple and unipotent commuting operators such that $(g_s)_{|K[G_2]}(g_s)_{|K[G_2]} = g_{|K[G_2]}$. Now our statement follows from the uniqueness of the Jordan decomposition.

Now consider the general case $\phi : G_1 \to G_2$. Since $\phi = \phi' \circ i$ where $\phi' : G \to \operatorname{Im}\phi$ and $i : \operatorname{Im}\phi' \hookrightarrow G_2$ we can apply the previous results. □

Theorem 6.5. *Let $G \leq GL_n(K)$ be a closed subgroup and let $g \in G$. Then g_s, g_u are just semisimple and unipotent part of the Jordan decomposition of a linear operator in $GL_m(K)$.*

Proof. We may assume $G = GL_m(K) = GL(V), \dim V = m$ (by Theorem 6.4). Let $0 \neq f \in \mathrm{Hom}_K(V, K)$. For every $v \in V$ define $\tilde{f}(v) \in K[G]$:

$$\tilde{f}(v)(g) \stackrel{def}{=} f(gv).$$

Then

$$\tilde{f} : V \to K[G]$$

is a monomorphism of linear spaces. We have

$$\rho(g)\tilde{f}(v)(x) = \tilde{f}(v)(xg) = f(xgv), \ \ \tilde{f}(gv)(x) = f(xgv) \Rightarrow$$

$$\rho(g)\tilde{f}(v) = \tilde{f}(gv). \tag{7}$$

Thus,

$$\rho(g_s)\tilde{f}(v) = \tilde{f}(g_s v), \ \ \rho(g_u)\tilde{f}(v) = \tilde{f}(g_u v). \tag{8}$$

By Lemma 6.3., ii.,

$$\rho(g)_s\tilde{f}(v) = \tilde{f}((g_s)v), \ \ \rho(g)_u\tilde{f}(v) = \tilde{f}((g_u)v) \tag{9}$$

where $(g_s), (g_u)$ are semisimple and unipotent part of g as a finitely dimensional linear operator. From the definition of g_s, g_u and (7), (8) we have

$$\tilde{f}((g_s)v) = \tilde{f}(g_s v), \ \ \ \tilde{f}((g_u)v) = \tilde{f}(g_u v)$$

for every $v \in V$. Since $\tilde{f}$ is an injection we get

$$g_s = (g_s), \ \ g_u = (g_u).$$

□

Exercises.

1) Let G be an algebraic group and let G_s, G_u be subsets of semisimple and unipotent elements of G. Show that G_u is a closed subset of G but G_s is not necessary closed.

Remark 6.6. *It is well-know fact from the general theory of linear groups that every subgroup of $GL_n(K)$ which consists of unipotent matrices is conjugate to a subgroup of U_n which is a nilpotent subgroup. If $G = G_u$ then G is a nilpotent group.*

Remark 6.7. *Suppose $G = G_u$ for some algebraic group G. Let X be an affine G-space and O is an orbit of G. Put $Y = \overline{O}, Z = Y \setminus O$. Then Z is a closed subset of Y. Consider the action of G on G-stable subset of $\mathfrak{I}_Y(Z) \subset K[Y]$. By Lemma 5.3, we can find a finitely dimensional G- stable subset of $\mathfrak{I}_Y(Z)$ and therefore there exists a non-zero G-stable function in $\mathfrak{I}_Y(Z)$. Hence f is constant on O and therefore on Y . Since $f(z) = 0$ for every $z \in Z$ we have $\mathfrak{I}_Y(Z) = K[Y]$ and $Y = O$. Thus, every orbit of an unipotent algebraic group acting on affine varieties is closed.*

7. Commutative groups.

Remark 7.1. *Let G, H be two algebraic groups. Then the product of affine varieties $G \times H$ is also algebraic group with multiplication $(g_1, h_1)(g_2, h_2) = (g_1g_2, h_1h_2)$ (check it!). Now if $G, H \leq F$ be subgroups of an algebraic group F then the map*

$$(g, h) \to gh$$

is always a morphism of varieties. If , in addition, G commutes with H it is a morphism of algebraic groups.

Theorem 7.2. *Let G be a commutative algebraic group. Then G_s, G_u are closed subgroups and the natural homomorphism*

$$G_s \times G_u \to G$$

is an isomorphism of algebraic groups.

Proof. We omit the proof of the following Lemma.

Lemma 7.3. *i. Two commuting semisimple (unipotent) linear operators are again semisimple (unipotent).*

ii. The set of commuting semisimple linear operators are conjugate to a subset of D_n.

iii. The set commuting linear operators a conjugate to a subset of T_n.

We may assume $G \leq GL(V) = GL_n, n = \dim V$.

Lemma 7.3. implies that G_s, G_u are subgroups of G. By Exercise 1 of the section 6, G_u is closed in G.

Again Lemma 7.3,ii, implies $G_s = G \cap gD_n g^{-1}$ for some $g \in GL_n$. Hence G_s is closed.

The Jordan decomposition give us a group isomorphism

$$m : G_s \times G_u \to G$$

$(m((s, u)) = su)$ which is also a morphism of varieties.

We may assume $G \subset T_n$ (Lemma 7.3,iii). Hence the map

$$g \to g_s$$

is a morphism of varieties $G \to G_s$. Hence

$$g \to (g_s, g_s^{-1} g)$$

is a morphism of varieties.

□

Remark 7.4. *If an algebraic group is connected and has the dimension one it is always commutative and coincides with G_s or G_u (see,* [Sp], *3.13).*

Algebraic torus.

Definition 7.5. *An algebraic group is called diagonalizable if it is isomorphic to a closed subgroup of D_n for some n. An algebraic group is called an algebraic torus if it is isomorphic to D_n for some n*

Definition 7.6. *The morphism of algebraic groups $\chi : G \to G_m$ is called a rational character of G.The morphism of algebraic groups $\chi : G_m \to G$ is called a rational co-character of G. The set of all rational characters (co-characters) are denoted by $X^*(G)$ and ($X_*(G)$).*

Remark 7.7. *The set $X^*(G)$ is always a commutative group with respect to usual multiplications of functions. The set $X_*(G)$ is also abelian group in the case where G is abelian.*

Remark 7.8. *By Dedekind Theorem , the characters $Hom(A, K^*)$ of an abelian group A are linearly independent over K (*[L]*, VIII, § 4).*

Example 7.9. *Let $G = D_n$. Then the map $\chi_i : G \to G_m$ such that $\chi(\mathrm{diag}(\alpha_1, \dots, \alpha_n)) = \alpha_i$ is a rational character of G and $K[G] = K[\chi_1, \dots, \chi_n, \chi_1^{-1}, \dots, \chi_n^{-1}]$ (check it !). Moreover, $X^*(G) \approx Z^n$.*

We omit the proof of the following results.

Theorem 7.10. *Let G be an algebraic groups. Then the following properties are equivalent:*

a. G is diagonalizable;

b. $X^*(G)$ is an abelian group of finite type and its elements are a basis of $K[G]$;

c. any rational representation is a direct sum of one dimensional rational representations.

Theorem 7.11. *Let G be a diagonalizable algebraic group. Then*

1. $K[G] = K[X^*(G)]$ (here $K[X^*(G)]$ is the group algebra of $X^*(G)$;

2. $G = T \times A$ where T is a torus and A is a finite abelian group of order m, such that $(m, \mathrm{char}\, K) = 1$;

3. the set of all elements of finite order is a subgroup of G and this subgroup is dense in G;

4. G is torus $\Leftrightarrow$ G is connected $\Leftrightarrow$ $X^*(G)$ is a free abelian group.

Theorem 7.12. *Let T be a torus and let $S \leq T$ be a connected closed subgroup then S is a torus and $T = S \times S_1$ for some subtorus $S_1 \leq T$.*

Theorem 7.13. *Let $K \neq \overline{F_p}$ and let T be a torus. Then there exists an element $t \in T$ such that $\overline{\langle t \rangle} = T$.*

Remark 7.14. *Let T be F-defined torus where F is subfield of K. We say that T is split over F if T is F-isomorphic to D_n (it means that there is an isomorphism of F-algebras $F[T]$ and $F[D_n] = F[t_1, \dots, t_n, t_1^{-1}, \dots, t_n^{-1}]$ where $F[T]$ are F-structures on G . A torus T defined over F is split if and only if all characters are defined over F.*

Example 7.15. *Let $A = \mathbb{R}[x,y]/(x^2+y^2-1)$ and let $B = A \otimes_{\mathbb{R}} \mathbb{C}$.*

Then $B \approx \mathbb{C}[t,t^{-1}]$ where $t = x+iy$ and $T = \operatorname{Spec}_{\mathfrak{m}} B$ is one-dimensional torus. Since A is not isomorphic to $\mathbb{R}[t,t^{-1}]$ the $\mathbb{R}$-defined torus T is not split. Also, since T is one dimensional torus, $X^*(T) \approx \mathbb{Z}$, but only a trivial character is defined over $\mathbb{R}$.

8. Soluble subgroups

Remark 8.1. *Let G be an algebraic group and let $N \trianglelefteq G$ be a closed normal subgroup. In this case there exists a rational representation $\rho : G \to GL(V)$ such that $\operatorname{Ker}\rho = N$ ([H], 11.5). Thus we may consider a factor-group G/N as linear algebraic group.*

Theorem 8.2. *(Lie-Kolchin). Let $G \leq GL(V)$ be a closed connected soluble subgroup. Then $gGg^{-1} \subset T_n$ for some $g \in GL(V)$.*

Proof.

Lemma 8.3. *There exists a vector $u \in V$ and a character $\epsilon \in X^*(G)$ such that $g(u) = \epsilon(g)u$ for very $g \in G$*

Proof. By Corollary 2.9, we have $[G,G]$ is a closed connected subgroup. Since G is soluble then $[G,G]$ is soluble and $[G,G] \neq G$. Hence $\dim[G,G] < \dim G$.

We will prove the assertion by induction on dimension starting with $\dim G = 0$. Suppose, the assertion holds for $[G,G]$. Then there exits a vector $v \in V$ such that $g(v) = \chi(g)v$ for every $g \in G$ and some $\chi \in X^*(G)$. For every $\psi \in X^*(G)$ put

$$V_\psi = \{v \in V \quad | \quad g(v) = \psi(g)v \ \text{ for every } \ g \in [G,G]\}.$$

Then G permutes non-zero spaces V_ψ. Since G is connected this permutation is trivial. Hence $g(V_\psi) = V_\psi$ for every ψ. Let us fix some non-zero V_ψ and consider the rational representation

$$\rho : G \to GL(V_\psi)$$

given by the formula $\rho(g) = g_{|V_\psi}$. Let $F = \rho(G)$. Then $[F,F] = \rho([G,G])$. Hence $f(v) = \psi(f)v$ for every $v \in V_\psi$ and every $f \in [F,F]$. Further, for every $f \in [F,F] \leq GL(V_\psi)$ we have $\det f = 1$. Hence $[F,F]$ is a finite group. But $[F,F]$ is a connected group (because $F = \rho(G)$ is connected) and therefore F is an abelian group. Now by Lemma 7.3, iii., we have a vector $u \in V_\psi$ such that $f(u) = \theta(f)u$ for every $f \in F$ and some $\theta \in X^*(F)$. Hence $g(u) = \epsilon(g)$ for every $g \in G$ and some $\epsilon \in X^*(G)$. □

Now the statement of the Theorem easily follows from the Lemma. □

Example 8.4. *Two-dimensional soluble finite group in characteristic zero are not triagonalizable.*

By a *maximal torus* of an algebraic group G we mean a subgroup $T \leq G$ which is a maximal torus in G with respect to inclusion.

In the following theorem we collect (without proofs) some important properties of connected soluble groups.

Theorem 8.5. *Let G be a soluble connected algebraic group. Then*

1. The set G_u is a closed connected nilpotent subgroup and the factor-group is a torus.

2. The commutator subgroup $[G, G]$ is closed connected subgroup of G_u.

3. Every semisimple element $s \in G$ is contained in a maximal torus and its centralizer $Z_G(s)$ is a connected subgroup of G.

4. Two maximal tori of G are conjugate and have the dimension $\dim G/G_u$.

5. If T is a maximal torus of G then the map of multiplication

$$\mu : T \times G_u \to G$$

is an isomorphism of affine varieties.

9. Parabolic subgroups

Recall, that an algebraic variety X is called a *complete* variety if for any variety Y the projection $X \times Y \to X$ is a *closed morphism*, i.e. maps the closed set onto closed sets. All projective varieties are complete. Below, we formulate some known properties of complete varieties ([Sp], 6.12). Below X is a complete variety.

a. A closed subset of X is complete.

b. If Y is a complete variety, then $X \times Y$ is a complete.

c. If $\phi : X \to Y$ is a morphism of varieties, then $\operatorname{Im} \phi$ is a complete variety.

d. If X is irreducible then any regular function on X is a constant;

e. If X is affine then X is a finite set of points.

Recall, that for any closed subgroup H of an algebraic group G we can construct an algebraic variety X with a fixed point $x \in X$ which is a G-space consisting of one orbit and such that the morphism $\pi : G \to X$ given by the formula $\pi(g) = g(x)$ has fibers γH (Corollary 5.9). This variety we will call a quotient of G with respect to H and we will denote it by G/H.

Definition 9.1. *A closed subgroup P of an algebraic group G is called a parabolic subgroup if the quotient variety G/P is a complete.*

Remark 9.2. *Using the fact that the quotient is a quasi-projective variety one can check that the complete variety G/P is a projective variety.*

We omit the proof of the following simple facts

1) If P is parabolic in G and Q is parabolic in P then Q is parabolic in G.

2) If $P \leq Q \leq G$ are closed subgroups and P is parabolic in G then Q is also parabolic in G.

3) P is parabolic in $G \Leftrightarrow P^0$ is parabolic in G^0.

Theorem 9.3. *A connected algebraic group G contains a proper parabolic subgroup if and only if G is not soluble.*

Proof. We omit the proof of the following fact (see, [Sp],Lemma 6.2.1)

Lemma 9.4. *Let G be an algebraic group $\psi : X \to Y$ be a morphism of G-spaces (which is a morphism of varieties such that $\psi(g(x)) = g\psi(x)$ for every $x \in X$ and every $g \in G$). Suppose that X and Y are just orbits and ψ is the bijection. Then X is complete if and only if Y is complete.*

Let $G \leq GL(V)$. Then G acts on $P(V)$. Then we can find a closed orbit $X \subset P(V)$ (Proposition 4.2) which is a projective variety. Hence X is a complete variety. There exists a bijective morphism $G/P \to X$ for some closed subgroup (let $x \in X$ then $P = G_x = \{g \in G \mid g(x) =$

$x\}$). By Lemma 9.4, the quotient G/P is complete. Suppose $G = P$ then $X = \{x\}$ and we can consider the action of G on V/x. Thus, acting in the same way we either find a proper parabolic subgroup or we construct a basis in V where G is triangalizable.

Now let G is a soluble group. Let us show that there is no a proper parabolic subgroup of G using induction on the dimension. Let $P \leq G$ be a proper parabolic subgroup of maximal dimension. By property 3) above, we may assume P is a connected. Then the group $Q = \langle P[G,G] \rangle$ is a connected closed subgroup containing P (Theorem 2.8). Hence Q is parabolic (by property 2)). We have two possibilities $Q = G$ or $Q = P$. Let $Q = G$. Then we have a morphism of $[G,G]$-varieties which is a bijection

$$[G,G]/[G,G] \cap P \to G/P.$$

By Lemma 9.4, $[G,G] \cap P$ is a parabolic subgroup of $[G,G]$. By assumption of induction $[G,G] \cap P = [G,G]$ and therefore $[G,G] \subset P$. Hence $Q = P \neq G$ which is a contradiction with assumption $Q = G$. Let $Q = P$. Then $[G,G] \leq P$ and therefore $P \lhd G$. Then the quotient G/P is an affine variety (see, Remark 8.1). Since G/P is a connected complete variety G/P is just one point (see, e. above). Thus, $G = P$. It is a contradiction with the assumption $P \neq G$.

□

Corollary 9.5. *Let G be a connected soluble group and let X be a complete G-space. Then there exists a G-stable point of X.*

Proof. Let $O \subset X$ be a closed orbit. Then the stabilizer of a point $o \in O$ is a parabolic subgroup of G and by Theorem 9.5. it coincides with the whole group G. □

Definition 9.6. *Let G be an algebraic group. A maximal closed connected soluble subgroup B of G is called a Borel subgroup.*

We omit the proof of the following theorem ([Sp], 6.2.7)

Theorem 9.7. *Let G be an algebraic group. Then*

i. a closed subgroup of G is a parabolic if and only if it contains a Borel subgroup;

ii. two Borel subgroups are conjubate;

iii. a Borel subgroup is parabolic;

iv. if $G \to H$ is a surjective morphism of algebraic groups then the image of a Borel subgroup in G is a Borel subgroup in H.

v. if G is connected then normalizer $N_G(P) = P$ for every parabolic subgroup $P \leq G$.

Example 9.8. *Let $G = GL_n$. Then $B = T_n$ is a Borel subgroup.*

Definition 9.9. *Let G be an algebraic group a torus $T \leq G$ is called maximal if it is not strictly contained in another subtorus $S \leq G$.*

Theorem 9.10. *Two maximal tori of an algebraic group are conjugate.*

Proof. Let T be a maximal torus of an algebraic group G. Then $T \leq B$. Since every pair of Borel subgroups are conjugate (9.7,ii.)and since every pair of maximal torus in a soluble group are conjugate (8.5., 4)) we get the statement. □

Definition 9.11. *Let G be an algebraic group and let $T \leq G$ be a maximal torus. The group $Z_G(T)^0$ is called a Cartan subgroup of G.*

Remark 9.12. *Let G be a connected algebraic group and $T \leq G$ be a torus. Then the centralizer $Z_G(T)$ is also a connected group (*[Sp], *6.4.7) . If, in addition, T is a maximal torus the group $Z_G(T)$ is nilpotent (*[Sp], *6.4.2)*

We omit the proof of the following theorems.

Theorem 9.13. *Let G be a connected algebraic group. Then*

1. Every element of G is contained in a Borel subgroup.

2. Every semisimple element is contained in a maximal torus.

3. The union of the Cartan subgroups contains a dense open subset of G

Example 9.14. *Let $G = GL_n$. Then every parabolic subgroup is conjugate to a standard parabolic subgroup*

$$P = \{ \begin{pmatrix} L_1 & * & * & * & * \\ \mathbf{0} & L_2 & * & \cdots & * \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & L_m \end{pmatrix} \quad | \quad L_i \in GL_{n_i}, \quad n_1 + n_2 + \cdots + n_m = n \}.$$

Any Cartan subgroup of G coincides with a maximal torus, i.e. with a group gD_ng^{-1}, $g \in GL_n$. Thus the union of Cartan subgroups is a set of all semisimple elements of G.

10. Lie algebra of an algebraic group

Let X be an affine algebraic variety and let $x \in X$. Recall, that the tangent space of X at the point x is the linear space

$$T_x = \mathrm{Hom}_K(\mathfrak{M}_x/\mathfrak{M}_x^2, K) \approx \mathrm{Der}_x(K[X], K) =$$

$$= \{D \in \mathrm{Hom}_K(K[X], K],$$

$$| \quad D(fg) = f(x)D(g) + g(x)D(f), D(\alpha) = 0, \ \alpha \in K\}.$$

If $\phi : X \to Y$ be a morphism of varieties then there is induced linear map

$$d_{x,\phi} : T_x \to T_{\phi(x)}$$

which is called a differential at the point x. (I, Section 2.)

Now let G be an algebraic group and let $A = K[G]$. Then we have two linear representation $\lambda, \rho : G \to A$ given by the formulas $\lambda(g)(f)(x) = f(g^{-1}x)$, $\rho(g)(f)(x) = f(xg)$. We can extend these representations to the derivation algebra $\mathrm{Der}_K(A, A)$: put for $D \in \mathrm{Der}_K(A, A)$:

$$\lambda(g)D \overset{def}{=} \lambda(g) \circ D \circ \lambda(g)^{-1}, \ \ \rho(g)D \overset{def}{=} \rho(g) \circ D \circ \rho(g)^{-1}.$$

(Indeed, have

$$\lambda(g)D(fh) = \lambda(g) \circ D \circ \lambda(g)^{-1}(fh) =$$

$$= \lambda(g)(\lambda^{-1}(g)f\, D(\lambda^{-1}(g)h) + \lambda^{-1}(g)h\, D(\lambda^{-1}(g)f)) =$$

$$= f\lambda(g)(D(\lambda^{-1}(g)h)) + h\lambda(g)(D(\lambda^{-1}(g)f))) =$$

$$= f\lambda(g)D(h) + h\lambda(g)G(f).$$

In the same way we check it for ρ.) The set of $\lambda(G)$-invariant derivations

$$\mathrm{Der}_K(A,A)^{\lambda(G)} = \{D \in \mathrm{Der}_K(A,A) \quad | \quad \lambda(g)D = D \text{ for ever } g \in G\}$$

we will denote by the symbol $L(G)$. The set $L(G)$ is a linear space over K. Moreover, we can introduce an algebraic operation on $L(G)$ which is called Lie bracket

$$[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1.$$

This operation give us the structure of a Lie algebra on $L(G)$.

Definition 10.1. *The Lie algebra $L(G)$ is called the Lie algebra of the group G.*

We omit the proof of the following theorem.

Theorem 10.2. *i. The map*

$$\alpha_G : L(G) \to T_e \approx \mathrm{Der}_e(A,K)$$

given by the formula $\alpha_G(D)(f) = D(f)(e)$ is an isomorphism of linear spaces.

ii. The morphism of algebraic groups $\phi : G_1 \to G_2$ induces the homomorphism of Lie algebras

$$\alpha_{G_1}^{-1} \circ d_{e,\phi} \circ \alpha_{G_1} : L(G_1) \to L(G_2).$$

Corollary 10.3.

$$\dim L(G) = \dim G.$$

Proof. All points of G are regular (Theorem 2.1). Hence $\dim T_e = \dim G$. □

Remark 10.4. *Now we will identify the linear space of Lie algebra $L(G)$ with T_e.*

Let G be an algebraic group and let $L(G)$ be its Lie algebra. Since $\rho(g)$ commutes with $\lambda(g)$ the $\lambda(G)$-invariant subspace $L(G) = \mathrm{Der}_K(A,A)^{\lambda(G)}$ is $\rho(G)$ stable. Hence we have a rational representation:

$$Ad : G \to GL(L(G))$$

$Ad(x)D = \rho(g)D$ which is called the *adjoint representation* of G.

Now let $H \leq G$ be a closed subgroup of G and let $I = \mathfrak{I}_G(H)$. Consider

$$\mathfrak{D}_{G,H} \overset{def}{=} \{D \in L(G) \quad | \quad DI \subset I\}.$$

Obviously, $\mathfrak{D}_{G,H}$ is a Lie subalsgebra of $L(G)$. Then we have a natural homomorphism

$$\nu : \mathfrak{D}_{G,H} \to L(H)$$

given by the formula

$$\nu(D)(f + I) = D(f) + I.$$

One can check that it is an isomorphism of Lie algebras. Thus for a closed subgroup $H \leq G$ we can identify the Lie algebra $L(H)$ with a subalgebra of $L(G)$.

Example 10.5. *1) Let $G = GL_n$. Then $A = K[G] = K[\{x_{ij}\}, det^{-1}]$. Put $\mathfrak{gl}_n = M_n(K)$. Then $\mathfrak{gl}_n$ is the Lie algebra with Lie bracket $[X,Y] = XY - YX$. Let $X = (X_{ij}) \in \mathfrak{gl}_n$. Then*

$$D_X(x_{ij}) = -\sum_{k=1}^{n} X_{ik}x_{kj}$$

is a derivation of $K[G]$ which is $\lambda(G)$-invariant. Thus we have a map $X \to D_X \in L(G)$ which is an isomorphism of Lie algebras $\mathfrak{gl}_n$ and $L(G)$. Here

$$Ad(g)X = gXg^{-1}.$$

2)Let $G = SL_n$. Then $\mathfrak{sl}_n = \{X \in \mathfrak{gl}_n \quad | \quad trx = 0\}$.

Some useful formulas.

a. $d_{e,\mu}(X,Y) = X + Y$;
b. $d_{e,i}(X) = -X$;
c. $d_{e,Ad}(X)(Y) = XY - YX = [X,Y]$;
d. $d_{e,[a,x]}(X) = (Ada - 1)X$.

11. Reducive algebraic groups

Let G be an algebraic group and let $H_1, H_2 \trianglelefteq G$ be two closed connected normal subgroup. Then the group $H = \langle H_1, H_2 \rangle$ is a closed connected (Theorem 2.8) and normal subgroup of G. Thus there exists the maximal (with respect to inclusion) closed connected normal soluble subgroup of G and the maximal closed connected normal unipotent subgroup of G

Definition 11.1. *The maximal closed connected normal soluble subgroup $R(G)$ of an algebraic group G is called the radical of G The maximal closed connected normal unipotent subgroup $R_u(G)$ of G is called the unipotent radical of G. An algebraic group G is called reductive if $R_u(G) = \{e\}$.An algebraic group G is called semisimple if $R(G) = \{e\}$.*

Let G be a connected reductive group. Then $H = R(G)$ is a torus (Theorem 8.5) which is contained in the center of G because for any torus $H \leq G$ we have $\mid N_G(H)/Z_G(H) \mid \leq \infty$ ([Sp], 3.2.9). The group $G/R(G)$ is a semisimple group.

Definition 11.2. *The dimensional of a maximal torus of a reductive group G is called rank G and we denote it by* $\operatorname{rank} G$. *The rank of $G/R(G)$ is a semisimple rank of G and we denote it by* $\operatorname{rank}_s G$.

Below G is a connected reductive group.

Weights .

Let $r : G \to GL(V)$ be a rational representation and let $T \leq G$ be torus . Then

$$V = \sum_{chi} V_\chi$$

where $\chi \in X^*(T)$ and

$$V_\chi = \{v \in V \mid t(v) = \chi(t)v\}$$

(Theorem 7.10). The character χ is called *a weight* of T in V. Vector-spaces V_χ are called *weight spaces* of T in V and their non-zero vectors are called *weight vectors*.

Now let $G \leq GL(V)$ and let $H = R(C)$ and let

$$V = \sum_{\psi} V_{\psi}, \quad \psi \in X^*(H)$$

be the decomposition into the sum of weight-spaces of torus H. Since $H \lhd G$ the group G interchange V_ψ and hence stabilizes each V_ψ (because G is a connected group.) Let $d \in [G,G] \cap H$. Then the restriction of d on V_ψ is a scalar operator which has determinant one. It implies

$$| \, [G,G] \cap R(G) \, | < \infty.$$

Hence $[G,G]$ is a semisimple group and

$$G = [G,G]R(G).$$

Weights of maximal torus in Lie algebra.

Let $T \leq G$ be a maximal subtorus. We consider rational representation

$$Ad : G \to GL(L(G))$$

and its restriction

$$r = Ad_{|T} : T \to GL(L(G)).$$

We have

$$L(G) = \sum_{\chi \in \mathfrak{X}} L_\chi$$

where L_χ is weight space corresponding to a rational character $\chi \in \mathfrak{X} \subset X^*(T)$.

Example 11.3. *Let $G = GL_n, SL_n$. Then a weight space of $\mathfrak{gl}_n, \mathfrak{sl}_n$ corresponds to a linear space of matrices where there are non-zero elements only on a fixed i, j entry $i \neq j$ or any one dimensional linear space of diagonal matrices.*

We put for every $\alpha \in \mathfrak{X}$

$$G_\alpha = Z_G((\operatorname{Ker} \alpha)^0).$$

Then groups $\{G_\alpha\}_{\alpha \in \mathfrak{X}}$ generate the group G. Moreover, G is soluble if and only if every group G_α is soluble (see, [Sp], 7.1.3). Now we put

$$\mathfrak{X}' = \{\alpha \in \mathfrak{X} \;\; | \;\; G_\alpha \text{ is not soluble } \}.$$

Further, the group

$$W(G,T) = N_G(T)/Z_G(T)$$

is finite ([Sp], 3.2.9) and acts by permutation on $\mathfrak{X}, \mathfrak{X}'$. It is called *the Weyl* of (G,T). Obviously, the Weyl group can be considered as a group of automorphisms of the whole group of characters $X^*(T)$.

If $\alpha \in \mathfrak{X}'$ then $G_\alpha/(\operatorname{Ker}\alpha)^0 \approx SL_2, PGL_2$ ([Sp], 7.1.5, 7.2.4.). Moreover,

$$W_\alpha = W(G_\alpha, T) = \langle e, w_\alpha \quad | \quad w_\alpha^2 = e\rangle$$

(here e is the identity of the group.)

Example 11.4. *Let $G = GL_n, SL_n$. Let $\alpha_{ij} \in \mathfrak{X}$ be a weight corresponding to the linear space of matrices M_{ij} where only in $i,j,\ i \neq j$ entry are non-zero elements. Then*

$$G_{\alpha_{ij}} = T\langle E + m_{ij}, E + m_{ji} \quad | \quad m_{ij} \in M_{ij}, m_{ji} \in M_{ji}\rangle.$$

Moreover,

$$\langle E + m_{ij}, E + m_{ji} \quad | \quad m_{ij} \in M_{ij}, m_{ji} \in M_{ji}\rangle \approx SL_2(K).$$

Further, the character corresponding to a one-dimensional subspace of diagonal matrices is trivial. Thus, $\alpha_0 \in \mathfrak{X},\ \alpha_0(T) = 1$. We have $G_{\alpha_0} = T$.

Further, $Z_G(T) = T$ and $W(G,T) \approx S_n$.

Example 11.5. *Consider case $G = SL_2, PGL_2$. Then*

$$L(G) = \mathfrak{sl}_2 = H + \mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}$$

where

$$\mathfrak{h} = \{\operatorname{diag}(s,-s),\},\ \ \mathfrak{g}_\alpha = \{\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\},\ \ \mathfrak{g}_{-\alpha} = \{\begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}\}.$$

Now let

$$h_\alpha = \{\operatorname{diag}(1,-1),\},\ \ v_\alpha = \{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\},\ \ v_{-\alpha} = \{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\}.$$

Then

$$[h_\alpha, v_\alpha] = 2v_\alpha,\ \ [h_\alpha, v_{-\alpha}] = -2v_{-\alpha}, [v_\alpha, v_{-\alpha}] = h_\alpha.$$

Recall, $X^*(T) \approx \mathbb{Z}^n$, $n = \dim T$. Put

$$X = X^*(T), \ \ X^\vee \overset{def}{=} \mathrm{Hom}(X, \mathbb{Z}), \ \ E = X \otimes \mathbb{R}, \ \ E^\vee = X^\vee \otimes \mathbb{R}.$$

We have a non-degenerate pairing

$$\langle \ , \ \rangle : X \times X^\vee \to \mathbb{Z}$$

given by the formula

$$\langle \chi, \psi \rangle = \psi(\chi)$$

which is induces a non-degenerate pairing

$$\langle \ , \ \rangle : E \times E^\vee \to \mathbb{R}.$$

The action of the Weyl group $W = W(G,T)$ on $X = X^*(T)$ and the induced action on $X^\vee$ we can extend on $E, E^\vee$. Moreover, we can chose W-invariant positively defined bilinear form $(\ , \ )$ on E. Then for every $\alpha \in \mathfrak{X}'$ the linear operator w_α on E is a reflection and

$$w_\alpha(v) = v - \frac{2(v,\alpha)}{(\alpha,\alpha)}\alpha = v - \langle v, \alpha^\vee \rangle \alpha$$

for some $\alpha^\vee \in X^\vee$ such that $\langle \alpha, \alpha^\vee \rangle = 2$. ([Sp],7.1.8). We also have

$$W = \langle w_\alpha \ \ | \ \ \alpha \in \mathfrak{X}' \rangle.$$

Root datum.

Now we may formalize the construction given above. Let X be a free abelian group of finite rank and let $X^\vee$ be a dual group with respect to pairing $\langle \ , \ \rangle : X \times X^\vee \to \mathbb{Z}$. Let $R \subset X, R^\vee \subset X^\vee$ be two fixed finite sets and let a bijection $\alpha \in R \to \alpha^\vee \in R^\vee$ is fixed. We say that $\Psi = \Psi(X, R, X^\vee, R^\vee)$ is a *root datum* if the following axioms hold:

RD1. $\langle \alpha, \alpha^\vee \rangle = 2$ for every $\alpha \in R$;

RD2. if $\alpha \in R$ and $s_\alpha, s_\alpha^\vee$ are automorphisms of $X, X^\vee$ respectively defined by formulas

$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha, \ \ s_\alpha^\vee(y) = y - \langle \alpha, y \rangle \alpha^\vee$$

satisfy the following conditions

$$s_\alpha(R) = R, \ \ s_\alpha^\vee(R^\vee) = R^\vee.$$

From the axioms we can get $s_\alpha^2 = 1, s_\alpha^{\vee 2} = 1$ and $s_\alpha(\alpha) = -\alpha, s_\alpha^\vee(\alpha^\vee) = -\alpha^\vee$. The group generated by s_α we denote by $W(\Psi)$ and call it *the Weyl group of the root datum* Ψ.

Now let $Q = \langle R \rangle$ and let $V = Q \otimes \mathbb{R}$. Then axioms $RD1, RD2$ implies that R is a root system of V(see,N. Bourbaki. Lie Groups and Lie Algebras. IV-VII) , that is,

1) $0 \notin R$ and $\langle R \rangle = V$;

2) if $\alpha \in R$ then $\langle \alpha, \alpha^\vee \rangle = 2$ and $s_\alpha(R) = R$;

3) $\langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$ for every $\alpha, \beta \in R$.

It is clear that $R^\vee$ is a root system in $V^\vee = Q^\vee \otimes \mathbb{R}$.

Further, we can chose positively defined $W(\Psi)$-invariant bilinear form $(\ ,\ )$ on V and we can find a vector $v \in V$ such that $(\alpha, v) \neq 0$ for every $\alpha \in R$. Then we put

$$R^+ = \{\alpha \in R \ \mid \ (\alpha, v) > 0\}, \ R^- = \{\alpha \in R \ \mid \ (\alpha, v) < 0\}.$$

We have the following properties of R^+

a.

$$0 \notin \widehat{R^+} = \{\sum r_\alpha \alpha \ \mid \ \alpha \in R^+, \, r_\alpha \in \mathbb{R}, \, r_\alpha \geq 0 \ \text{ and } \ r_\alpha > 0 \ \text{ for some } \ \alpha\};$$

b.

$$R = R^+ \cup R^-.$$

c.

$$\alpha, \beta \in R^+, \alpha + \beta \in R \Rightarrow \alpha + \beta \in R^+.$$

The set R^+ (respectively, R^-) is called the set of *positive* (respectively, *negative*) roots. In a set positive roots R^+ there exists a subset $\Pi = \{\alpha_1, \dots, \alpha_r\} \subset R^+$ such that $r = \dim V$ and elements from Π are linearly independent in V and for every $\alpha \in R^+$

$$\alpha = \sum_i n_i \alpha_i, n_i \geq 0.$$

The set Π is unique and it is called the *the simple root system* corresponding to R^+.

Root datum of reductive groups.

Let G be a connected reductive group, $T \leq G$ be a maximal torus, $T \leq B \leq G$ be a Borel subgroup. Put $R = \mathfrak{X}' \subset X = X^*(T)$ and $R^\vee = \{r^\vee \in X^\vee \ \mid \ r \in R\}$. Then $\Psi(G, T) = (X, R, X^\vee, R^\vee)$ is a root datum. Further, for every $\alpha \in R$ the group $G_\alpha = Z_G((\operatorname{Ker} \alpha)^0)$ is reductive and $G/(\operatorname{Ker} \alpha^0) = SL_2, PGL_2$. We have

$$L(G \cap B) \leq L(G_\alpha) = \mathfrak{H} + \mathfrak{sl}_2 = \mathfrak{H} + \mathfrak{h} + \mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha} \leq L(G).$$

The group $G \cap B$ is a Borel subgroup of G_α ([Sp], 6.4.7)containing T. The Lie algebra of B should contain either $\mathfrak{g}_\alpha$ or $\mathfrak{g}_{-\alpha}$. We may assume that $\mathfrak{g}_\alpha \in L(B)$. Now let $R^+(B)$ be the set of roots $\alpha \in R$ satisfying the condition $\mathfrak{g}_\alpha \in L(G_\alpha \cap B)$. Then

$$R^+ \stackrel{def}{=} R^+(B)$$

is the set of positive roots in the previous sense ([Sp], 7.4.6).

Further, for every $\alpha \in R$ there exists an isomorphism

$$u_\alpha : G_a = K^+ \to G$$

such that

$$\operatorname{Im} du_\alpha = \mathfrak{g}_\alpha$$

and the following conditions hold ([Sp], §8):

1) groups $U_\alpha = \langle u_\alpha(s) \mid s \in K \rangle$ and T generate G;
2) if $U = \langle U_\alpha \mid \alpha > 0 \rangle$ then $U = B_u = [B, B], B = TU$, ;
3) $G_\alpha = \langle U_{\pm\alpha}, T \rangle$;
4) $\langle U_\alpha, U_{-\alpha} \rangle \approx SL_2$ or PGL_2;
5) $t u_\alpha(s) t^{-1} = u_\alpha(\alpha(t)s)$ for every $s \in K, t \in T$;
6) for every two roots $\beta \neq \pm\alpha$

$$[u_\alpha(x), u_\beta(y)] = \prod_{i\alpha + j\beta \in R, i,j > 0} u_{i\alpha + j\beta}(c_{\alpha,\beta,i,j} x^i y^j)$$

where $c_{\alpha,\beta,i,j}$ are constants (which depends on root system and the order of multiplications) (**Chevalley Commutator Formula**);

7) the element $\dot{w}_\alpha = u_\alpha(1) u_{-\alpha}(1) u_\alpha(1)$ has the order 4 or 2 ; the group generated by $\dot{w}_\alpha$ and T coincides with $N_G(T)$ and $\dot{w}_\alpha = w_\alpha($ mod $T)$;

Also the following properties of reductive groups hold ([Sp], §8):

a.$\dim G = \operatorname{rank} G + \mid R \mid$;
b.$\dim B = \operatorname{rank} G + \frac{1}{2} \mid R \mid$;
c. if G is a semisimple then G is generated by U_α and $[G, G] = G = G_1 G_2 \cdots G_k$ where for every i the group G_i (which is called a *simple algebraic group*) is semisimple and $G_i / Z(G_i)$ is a simple group (as an abstract group), and for every i, j elements of G_i commute with elements of G_j and $G_i \cap G_j \leq Z(G)$; moreover $Z(G)$ is a finite group.

Case of semisimple groups.

Let $G = [G,G] = G_1G_2\cdots G_k$ be a semisimple algebraic group. The groups G_i are called *simple components* of G. If $T \leq G$ be a maximal torus of G then $T_i = T \cap G_i$ is a maximal torus of G_i and $T = T_1T_2\cdots T_k$. If $T \leq B$ is a Borel subgroup then $T_i \leq B_i = G_i \cap B$ is a Borel subgroup of G_i and $B = B_1B_2\cdots B_k$. Further,

$$L(G) = \prod_{i=1}^{k} L(G_i)$$

and $L(G_i)$ is an ideal of $L(G)$.

Further,

$$X = X^*(T) = \underbrace{X^*(T_1)}_{=X_1} \oplus \underbrace{X^*(T_2)}_{=X_2} \oplus \cdots \oplus \underbrace{X^*(T_k)}_{=X_k},$$

$$X^\vee = X_1^\vee \oplus X_2^\vee \oplus \cdots \oplus X_k^\vee,$$

$$R = R(G,T) = \underbrace{R(G_1,T_1)}_{=R_1} \cup \underbrace{R(G_2,T_2)}_{=R_2} \cup \cdots \cup \underbrace{R(G,T_k)}_{=R_k}, \quad R_i \cap R_j = \emptyset, \tag{$*$}$$

$$Q = \langle R \rangle = \underbrace{\langle R_1 \rangle}_{=Q_1} \underbrace{\oplus \langle R_2 \rangle}_{=Q_2} \oplus \cdots \underbrace{\oplus \langle R_k \rangle}_{=Q_k}.$$

If the properties (*) hold for an abstract root system R with $k > 1$ we say that this root system is *reducible*. If a root system is not reducible it is called *irreducible*. All irreducible root systems are classified (see,(see,N. Bourbaki. Lie Groups and Lie Algebras. IV-VII). They are labelled by symbols $A_r, B_r, C_r, D_r, E_6, E_7, E_8, F_4, G_2$ (the index is *the rank of system* which is equal to the dimension of the space $V = Q \otimes R$).

Thus,

$$G \text{ is simple} \quad \Leftrightarrow \quad R = R(G,T) \quad \text{is irreducible.}$$

Put

$$P \stackrel{def}{=} \{v \in V \quad | \quad \langle v, R^\vee \rangle \subset \mathbb{Z}\}.$$

Then P is a lattice in V (which is called *the lattice of weights* and $Q \subset X \subset P($ Q is called *the lattice of roots*. The group G is called *simply connected* if $X = P$ and *adjoint* if $X = Q$.

Example 11.6. *Let* $G = SL_2$*. Then* $\dim T = 1, X = X^*(T) = \langle\gamma\rangle$ *where* $\gamma(\mathrm{diag}(t, t^{-1})) = t$*. Further,* $X^\vee = \langle\gamma^\vee\rangle, \langle\gamma, \gamma^\vee\rangle = 1$*. Further,* $R = \{\alpha = 2\gamma, -\alpha = -2\gamma\}$ *(here we use additive notation for weights; thus* $2\gamma(\mathrm{diag}(t, t^{-1})) = t^2$*). We have* $\gamma^\vee = \alpha^\vee, P = X, Q = \langle\alpha\rangle \neq X, \mid P/X \mid = 2$.

Parabolic subgroups and Bruhat decomposition of connected reductive groups.

Let G be a connected reductive group and let $T \leq B \leq G$ be a maximal torus and a Borel subgroup containing T. Recall, that $W = N_G(T)/Z_G(T)$ is the Weyl group of G. For every $w \in W$ denote by $\dot{w} \in N_G(T)$ a pre-image of w. Then the set

$$B\dot{w}B = \{b_1\dot{w}b_2 \mid b_1, b_2 \in B\}$$

does not depend on the choice of the pre-image of w. This set is called the *Bruhat cell* corresponding to the element w of the Weyl group. If $w_1 \neq w_2$ are two elements from the Weyl group then

$$B\dot{w}_1B \cap B\dot{w}_2B = \emptyset.$$

We have

$$G = \bigcup_{w \in W} B\dot{w}B.$$

(**The Bruhat decomposition** of G.)

Now let $R = R(T, G)$ be a root system and let $R^+ = R^+(B)$ be a positive roots system, and let $\Pi \subset R^+$ be the symple root system. Further, let $X \subset \Pi$ and

$$W_X \stackrel{def}{=} \langle w_\alpha \mid \alpha \in X\rangle.$$

Then $W_X = W \Leftrightarrow X = \Pi$. The set

$$P_X = \bigcup_{w \in W_X} B\dot{w}B$$

is a subgroup of G which contains B,i.e. a parabolic subgroup (which is called a *standard parabolic subgroup*). Every parabolic subgroup of G is conjugate to a standard one. There exists a connected reductive subgroup $L_X \leq G$ (which is called a *Levy factor* of P_X) such that its Weyl subgroup is equal to W_X and

$$P_X = LR_u(P_X).$$

The Main Theorem.

Theorem 11.7. *Let $\Psi = (X, R, X^\vee, R^\vee)$ be a root data. Then there exits a unique (up to isomorphism) connected reductive group G with a maximal torus T such that the root datum of this group coincides with Ψ.*

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